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L^1-Approximation of Meromorphic Functions

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Let $S = \{z_k\}$ be a countably infinite set in the complex plane $\mathbb{C}$ possessing no limit points in $\mathbb{C}$. Let $\mathfrak{B}$ be the collection of functions, $f(z)$, analytic in $\mathbb{C}\backslash S$, possessing finite L^1 norm,

$$\|f\| = \iint_{\mathbb{C}} |f(z)|\, dx\, dy < \infty. \tag{1.1}$$

If $a \in S$ and $0 < |z-a| < \frac{1}{2}\min_{b \in S} |a-b|$, then the mean-value property of analytic functions shows that, for any $f \in \mathfrak{B}$,

$$f(z) = \frac{1}{\pi |z-a|^2} \iint_{|\zeta - z| \leqslant |z-a|} f(\zeta)\, d\xi\, d\eta.$$

Hence,

$$|f(z)| \leqslant \frac{\|f\|}{\pi |z-a|^2}.$$

From this estimate and (1.1), we conclude that f is meromorphic in $\mathbb{C}$, and that the singularities of f are at worst *first*-order poles at the points of S.

It follows from very general criteria of Bers [1, 2] that every function of the class $\mathfrak{B}$ can be approximated arbitrarily well in the L^1 norm by *rational* functions, $f_n(z)$. Bers' method makes use of Banach space theory, and distributional derivatives of singular integrals. Due to its general nature, it provides an existence theorem for the approximating sequence $\{f_n\}$, but not an explicit construction. By means of very elementary methods, we will see,

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below, how the sequence, f_n, can be obtained in a concrete manner in the case under discussion here.

For the construction of f_n, $n = 1, 2, \ldots$, we proceed as follows. Assume that $|z_{k+1}| \geqslant |z_k|$, for all $z_k \in S$, and let

$$F(r) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|\, r\, d\theta. \tag{1.2}$$

In view of (1.1), $F(r)$ exists for almost all $r > 0$, and

$$\|f\| = 2\pi \int_0^{\infty} F(r)\, dr < \infty.$$

Therefore, we may choose a sequence $\{r_n\}$, $\lim r_n = \infty$,

$$\max(2, |z_2|) < r_1 < r_2 < \cdots, \tag{1.3}$$

with $r_n \neq |z_k|$, $(z_k \in S)$, and such that

$$\lim_{n\to\infty} (r_n \log r_n)\, F(r_n) = 0. \tag{1.4}$$

Let λ_k denote the residue of f at z_k. We define $f_n(z)$ as

$$f_n(z) = \sum_{|z_k| < r_n} \frac{\lambda_k}{z - z_k} + \frac{\alpha_n}{z - z_1} + \frac{\beta_n}{z - z_2}, \tag{1.5}$$

where α_n, β_n are determined by the requirement that $z^2 f_n(z) \to 0$, as $z \to \infty$; namely, that

$$\begin{aligned} \sum_{|z_k| < r_n} \lambda_k + \alpha_n + \beta_n &= 0, \\ \sum_{|z_k| < r_n} z_k \lambda_k + z_1 \alpha_n + z_2 \beta_n &= 0. \end{aligned} \tag{1.6}$$

THEOREM. *Given $f \in \mathfrak{B}$, let f_n, $n = 1, 2, \ldots$, be chosen as in* (1.5). *Then*

$$\lim_{n\to\infty} \|f - f_n\| = 0.$$

Furthermore, $\lim f_n(z) = f(z)$, *uniformly in every compact subset of* $\mathbb{C} \setminus \{z_1, z_2\}$.

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We proceed with the proof.
Let

$$g_n(z) = \sum_{|z_k| < r_n} \frac{\lambda_k}{z - z_k}, \qquad h_n(z) = -\frac{\alpha_n}{z - z_1} - \frac{\beta_n}{z - z_2},$$

and

$$A_n = \frac{1}{2\pi i} \int_{|\zeta| = r_n} f(\zeta)\, d\zeta = \sum_{|z_k| < r_n} \lambda_k,$$
$$B_n = \frac{1}{2\pi i} \int_{|\zeta| = r_n} \zeta f(\zeta)\, d\zeta = \sum_{|z_k| < r_n} z_k \lambda_k.$$

As is easily verified,

$$\begin{aligned} g_n(z) &= f(z) + \frac{1}{2\pi i} \int_{|\zeta| = r_n} \frac{f(\zeta)}{z - \zeta}\, d\zeta, && |z| < r_n, z \neq z_k \\ &= \frac{1}{2\pi i} \int_{|\zeta| = r_n} \frac{f(\zeta)}{z - \zeta}\, d\zeta, && |z| > r_n \end{aligned} \tag{2.1}$$

Thus,

$$|g_n(z) - f(z)| \leqslant \frac{F(r_n)}{r_n - |z|}, \qquad |z| < r_n. \tag{2.2}$$

This shows that $g_n(z) \to f(z)$, uniformly in every compact subset of $\mathbb{C}$.
Solving (1.6) for α_n, β_n in terms of A_n, B_n, yields the formulas

$$\begin{aligned} h_n(z) &= \left(\frac{A_n z_2 - B_n}{z_2 - z_1}\right) \frac{1}{z - z_1} + \left(\frac{B_n - A_n z_1}{z_2 - z_1}\right) \frac{1}{z - z_2} \\ &= \frac{1}{2\pi i} \int_{|\zeta| = r_n} \frac{\zeta + z - (z_1 + z_2)}{(z - z_1)(z - z_2)} f(\zeta)\, d\zeta. \end{aligned} \tag{2.3}$$

Since

$$|A_n| \leqslant F(r_n), \qquad |B_n| \leqslant r_n F(r_n),$$

we see right away, from the first expression in (2.3) that

$$\lim_{n \to \infty} h_n(z) = 0,$$

uniformly in every compact subset of $\mathbb{C}\backslash\{z_1, z_2\}$. Hence the assertion about $\lim f_n(z)$ is proved. To prove the main assertion of the theorem we must verify that the norms $\|f_n\|$ are bounded unifromly with respect to n. (The choice of α_n, β_n was made to ensure that $\|f_n\| < \infty$.)

We consider first the quantity

$$\iint_{|z|<2r_n} |f_n(z)|\, dx\, dy.$$

By (2.1),

$$\begin{aligned}\iint_{|z|<2r_n} |g_n(z)|\, dx\, dy &\leqslant \iint_{|z|<r_n} |f(z)|\, dx\, dy \\ &\quad + \frac{1}{2\pi}\int_{|\zeta|=r_n} |f(\zeta)| \cdot |d\zeta| \iint_{|z|<2r_n} \frac{dx\, dy}{|z-\zeta|} \\ &\leqslant \|f\| + \frac{1}{2\pi}\int_{|\zeta|=r_n} |f(\zeta)| \cdot |d\zeta| \iint_{|z|<3r_n} \frac{dx\, dy}{|z|} \\ &= \|f\| + 6\pi r_n F(r_n).\end{aligned}$$

On the other hand, using the second expression for $h_n(z)$, in (2.3),

$$\begin{aligned}|h_n(z)| &\leqslant \frac{r_n + |z| + |z_1 + z_2|}{|z-z_1| \cdot |z-z_2|} F(r_n) \\ &\leqslant \frac{5r_n F(r_n)}{|z-z_1| \cdot |z-z_2|}, \qquad (|z| < 2r_n).\end{aligned}$$

Therefore,

$$\iint_{|z|<2r_n} |h_n(z)|\, dx\, dy = O[(r_n \log r_n)\, F(r_n)], \qquad n \to \infty.$$

Hence,

$$\iint_{|z|<2r_n} |f_n(z)|\, dx\, dy \leqslant \|f\| + O[(r_n \log r_n)\, F(r_n)], \qquad n \to \infty. \tag{2.4}$$

In order to estimate $\iint_{|z|>2r_n} |f_n(z)|\, dx\, dy$ we combine the second expression for $g_n(z)$ of (2.1) with the second expression for $h_n(z)$ of (2.3). The coefficient of $f(\zeta)$ in the resulting combined integrand is

$$\frac{1}{z-\zeta} + \frac{(z_1+z_2)-(\zeta+z)}{(z-z_1)(z-z_2)} = \frac{\zeta^2}{(z-\zeta)z^2} + \frac{z_1 z_2(z+\zeta)-(z_1+z_2)\zeta z}{(z-z_1)(z-z_2)z^2}.$$

Thus, for $|z| > r_n$,

$$|f_n(z)| \leqslant \frac{1}{2\pi |z|^2} \int_{|\zeta| = r_n} \left| \frac{\zeta^2}{z-\zeta} + \frac{z_1 z_2 (z+\zeta) - (z_1+z_2)\,\zeta z}{(z-z_1)(z-z_2)} \right|$$
$$\cdot |f(\zeta)| \cdot |d\zeta|$$
$$\leqslant \frac{1}{|z|^2} \left[\frac{r_n^2}{|z| - r_n} + \frac{|z_1 z_2| \,(|z| + r_n) + |z_1 + z_2|\, r_n |z|}{|z-z_1| \cdot |z-z_2|} \right] F(r_n),$$

whence

$$\iint_{|z| > 2r_n} |f_n(z)|\, dx\, dy = O[r_n F(r_n)], \qquad \text{as} \quad n \to \infty. \tag{2.5}$$

Together, (2.4) and (2.5) imply that

$$\|f_n\| \leqslant \|f\| + O[(r_n \log r_n)\, F(r_n)], \qquad n \to \infty.$$

In view of (1.4), therefore, we conclude that $\|f_n\|$ is bounded independently of n.

References

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